IOE 691: Approximation Algorithms Date: 03/08/2017

Lecture Notes: Multicut and Balanced Separator

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1 Multicut problem

The input is an undirected graph $G = (V, E)$ with non-negative edge costs c_e for all $e \in E$, source-sink pairs (s_i, t_i) for $i = 1, \ldots, k$. The goal is to find a minimum-cost subset of edges F such that for every $i \in [k]$,there is no path connecting s_i and t_i in $(V, E - F)$.

The main result is the following theorem

Theorem 1.1 *There is an $O(\log k)$-approximation algorithm for multicut problem.*

It is known that multicut is hard to approximate to any constant factor.

The algorithm is based on the following LP relaxation:

$$\begin{aligned}
\min \quad & \sum_e c_e x_e & \\
s.t. \quad & d(s_i, t_i) \geq 1, & \forall i \in [k] \\
& d(s, v) \leq d(s, u) + x_{uv}, & \forall (u, v) \in E \ \forall s \in V \\
& d(s, s) = 0, & \forall s \in V \\
& d \geq 0, x \geq 0 &
\end{aligned}$$

To see that this is a relaxation, consider the following integer solution for any cut $F \subseteq E$.

$$x_e = \begin{cases} 1 & \text{if edge } e \in F \\ 0 & \text{otherwise} \end{cases}, \quad d(s, v) = \begin{cases} 0 & \text{if vertex } v \text{ is connected to } s \text{ in } G \setminus F \\ 1 & \text{otherwise} \end{cases}$$

During the LP rounding, it will be convenient to think of $d(s, u)$ as the shortest path from s to u given edge length x_e. We also make the following observation.

Observation 1.1 *$d(s, v) \leq d_x(s, v)$, where $d_x(s, v)$ is the shortest s,v-path distance with edge length given by x. And we may assume $d(s, v) = d_x(s, v)$ at optimal LP.*

This is due to the constraints $d(s, v) \leq d(s, u) + x_{uv}$ and the fact that it is a minimization problem.

2 Region growing

Given an LP solution (x, d), we need a rounding algorithm. Simple extensions of the threshold-based rounding for min-cut only give an $\Omega(k)$-approximation.

For any vertex $s \in V$ and radius r, define the ball

$$B(s,r) = \{u \in V : d(s,u) \leq r\},$$

and the associated cut

$$\delta B(s,r) = \{(u,v) \in E : u \in B(s,r), v \notin B(s,r)\}.$$

Define (LP) volume of $B(s,r)$ as

$$V(B(s,r)) = \beta L^* + \sum_{u,v \in B} c_{uv} x_{uv} + \sum_{u \in B, v \notin B} c_{uv}(r - d_x(s,u)),$$

where $L^* = \sum_e c_e x_e$ is the optimal LP value (see also Figure 1) and β is a parameter we will set later. It can be seen in this figure that if an edge is completely contained in the ball, it will

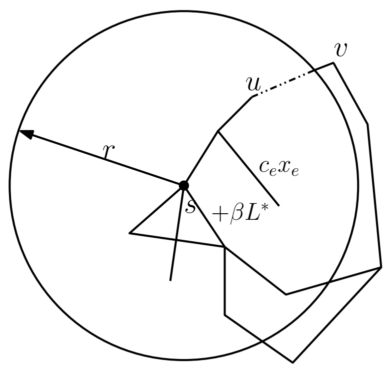

Figure 1: Volume of $B(s,r)$

contribute all its weight to the volume and if an edge is partially contained in the ball, it will contribute the proportion of its weight in the ball to the volume.

Lemma 2.1 (Region growing lemma (RGL) [1, 2]) *Consider any graph G with edge-costs c_e and edge-lengths x_e. Given any $s \in V$ and bound $b > 0$, a radius $r \leq b$ can be computed in polynomial time such that*

$$c(\delta B(s,r)) \leq V(B(s,r)) \frac{1}{b} \ln\left(\frac{\beta+1}{\beta}\right).$$

Proof: Let radius r increase from 0. Note that the cut value only changes when the radius touch some vertex. Hence only n choices of radius matter. Hence we only need to show there exists such radius satisfying the condition in the lemma. Consider the rate of volume change on radius, we have

$$\frac{dV(B(s,r))}{dr} = c(\delta(B(s,r)))$$

Prove by contradiction. Let $\rho = \frac{1}{b} \ln\left(\frac{\beta+1}{\beta}\right)$. Suppose for every r, $c(\delta B(s,r)) > \rho V(B(s,r))$. Then we have

$$\int_{V(0)}^{V(b)} \frac{dV}{V} > \rho \int_0^b dr$$

$$\Rightarrow \ln\left(\frac{V(b)}{V(0)}\right) > \rho b$$

where $V(r)$ is the volume of the ball with radius r.

But we know that $V(0) = \beta L^*$ and $V(b) \leq (\beta+1)L^*$ because $(\beta+1)L^*$ corresponds to infinite radius. Hence we have

$$\ln\left(\frac{(\beta+1)L^*}{\beta L^*}\right) \geq \ln\left(\frac{V(b)}{V(0)}\right) > \rho b = \ln\frac{\beta+1}{\beta}$$

which is a contradiction. ■

Rounding algorithm The rounding algorithm is

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Given graph G, initially F = ∅;
for i = 1, ..., k do
    if If s_i, t_i is still connected in G then
        Use RGL with s = s_i, b = 1/2 to get r and F_i = δB(s, r);
        F ← F ∪ F_i;
        Remove ball B(s_i, r) and incident edges from graph G;
    end
end
return F
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Algorithm 1: Rounding algorithm for multicut problem

Analysis We set $\beta = \frac{1}{k}$. This term is to ensure that the volume of any chosen ball is nonzero. And since in the proof we need to add up the volume of every vertices, we need β to be $O(\frac{1}{k})$ and we do not want a smaller β because it appear as $\ln(1+\frac{1}{\beta})$ in the approximation ratio.

Lemma 2.2 *F is a feasible solution for the multicut problem.*

Proof: The only possible way in which the solution might be infeasible is if we have some s_j,t_j pair in the ball $B(s_i, r)$ when the vertices in the ball get removed from the graph. We have the following claim:

Claim 2.1 *For any radius $r < \frac{1}{2}$ and any center $s_i \in V$, we have*

$$|B(s_i, r) \cap \{s_j, t_j\}| \leq 1, \quad \forall j$$

Prove by contradiction. Suppose both s_j and t_j are in $B(s_i, r)$. Then the distance between s_j and t_j is less than 1 by triangle inequality. This is a contradiction with constraint $d(s_i, t_i) \geq 1$. ■

Theorem 2.1 *The algorithm is a $(4\ln(k+1))$-approximation algorithm for the multicut problem.*

Proof: Let B_i be the set of vertices in the ball $B(s_i, r)$ chosen by the algorithm when pair s_i, t_i are selected for separation; we set $B_i = \emptyset$ if no ball is chosen for i. Let F_i be the edges in $\delta(B_i)$ when B_i and its incident edges are removed from the graph. Then $F = \bigcup_i F_i$. Let V_i be the total volume of edges removed when the vertices of B_i and its incident edges are removed from the graph. Note that $V_i \geq V(B(s_i, r)) - \frac{L^*}{k}$ since V_i contains the full volume of edges in F_i, which $V(B(s_i, r))$ contains only part of the volume of these edges but has an extra term of $\frac{L^*}{k}$. Note that by the choice of r in the region growing lemma, we have $c(F_i) \leq (2\ln(k+1))V(B(s_i, r)) \leq (2\ln(k+1))(V_i + \frac{L^*}{k})$.

Further, observe that the volume of each edge belongs to at most one V_i; once the edge is part of V_i, it is removed from the graph and cannot be part of the volume of a ball B_j removed in a later iteration. Putting all of this together, we have that

$$\sum_{e \in F} c_e = \sum_{i=1}^{k} \sum_{e \in F_i} c_e \leq (2 \ln(k+1)) \sum_{i=1}^{k} \left(V_i + \frac{L^*}{k} \right) \leq (4 \ln(k+1)) L^* \leq (4 \ln(k+1)) OPT.$$

■

3 Balanced separator problem

Given an undirected graph $G = (V, E)$ with edge costs c_e for $e \in E$, the goal is to find a minimum cost bisection S, i.e., $\min c(\delta(S))$ subject to $|S| = \frac{n}{2}$. Here $n = |V|$. The main result is the following theorem.

Theorem 3.1 *There is a bicriteria approximation algorithm for balanced separator that finds a solution $S \subseteq V$ with*

- $|S| \in \left[\frac{n}{5}, \frac{4n}{5}\right]$, *and*
- $c(\delta S) \leq O(\log n) OPT$, *where OPT is a bisection.*

We use the following LP relaxation. The variables are the same as multicut problem.

$$\begin{aligned}
\min \quad & \sum_{e} c_e x_e \\
& d(v,w) \leq d(v,u) + x_{uw}, && \forall (u,w) \in E,\ \forall v \in V \\
& \sum_{w \in V} d(v,w) \geq \frac{n}{2}, && \forall v \in V \\
& d(v,v) = 0, && \forall v \in V \\
& d \leq 1, \\
& x, d \geq 0
\end{aligned}$$

The second set of constraints ensure that each vertex is separated from $n/2$ other vertices, which is required in a bisection.

Rounding algorithm The ball, cut and volume of ball is defined same as the multicut problem. The rounding algorithm is

Given graph G, initially $U \leftarrow \emptyset$;
while $|U| < \frac{n}{5}$ **do**
    Pick any $s \notin U$;
    Apply RGL with s and $b = \frac{1}{6}$ to get $B(s,r)$;
    $U \leftarrow U \cup B(s,r)$;
    Remove ball $B(s,r)$ and incident edges from graph;
end
return U

Algorithm 2: Rounding algorithm for balanced separator problem

Analysis We set $\beta = \frac{1}{n}$. This term is to ensure that the volume of any chosen ball is nonzero. And since in the proof we need to add up the volume of every vertices, we need β to be $O(\frac{1}{n})$ and we do not want a smaller β because it appear as $\ln(1+\frac{1}{\beta})$ in the approximation ratio.

Claim 3.1 $|B(s, \frac{1}{6})| \le \frac{3}{5}n$

Proof: $\frac{n}{2} \le \sum_{v\in V} d(s,v) \le \frac{1}{6}|B(s,\frac{1}{6})| + n - |B(s,\frac{1}{6})|$. The second inequality comes from vertices in the ball have $d(s,v)$ at most $\frac{1}{6}$ and other vertices have $d(s,v)$ at most 1. ■

Claim 3.2 *Total cost* $c(\delta U) \le 12\ln(1+n)$ *total volume*

Proof: Let B_i be the set of vertices in the ball $B(i,r)$ chosen by the algorithm when vertex i is picked; we set $B_i = \emptyset$ if i is not picked. Let U_i be the edges in $\delta(B_i)$ when $B_i(i,r)$ and its incident edges are removed from the graph. Then $\delta U = \bigcup_i U_i$. Let V_i be the total volume of edges removed when the vertices of B_i and its incident edges are removed from the graph. Note that $V_i \ge V(B(i,r)) - \frac{L^*}{n}$ since V_i contains the full volume of edges in U_i, which $V(B(i,r))$ contains only part of the volume of these edges but has an extra term of $\frac{L^*}{n}$. Note that by the choice of r in the region growing lemma, we have $c(U_i) \le (6\ln(n+1))V(B(s_i,r)) \le (6\ln(n+1))(V_i + \frac{L^*}{n})$. Further, observe that the volume of each edge belongs to at most one V_i; once the edge is part of V_i, it is removed from the graph and cannot be part of the volume of a ball B_j removed in a later iteration. Putting all of this together, we have that

$$\sum_{e\in U} c_e = \sum_{i=1}^{n}\sum_{e\in U_i} c_e \le (6\ln(n+1))\sum_{i=1}^{n}\left(V_i + \frac{L^*}{n}\right) \le (12\ln(n+1))L^* \le (12\ln(n+1))OPT.$$

■

Now combined the above two claims, we can prove Theorem 3.1.

Proof: The solution cost is proved in Claim 3.2. Since the iterations continue as long as $|U| < \frac{n}{5}$ and increase in $|U|$ is at most $\frac{3}{5}n$ in each iteration, we have $|U| \in [\frac{n}{5}, \frac{4n}{5}]$. ■

References

[1] Gary, N., Vazirani, V., and Yanakakis, M. Approximation max-ow min-(multi) cut theorems and their applications,". In *Proc. of the ACM Symposium on Theory of Computing* (1993).

[2] Leighton, T., and Rao, S. An approximate max-flow min-cut theorem for uniform multicommodity flow problems with applications to approximation algorithms. In *Foundations of Computer Science, 1988., 29th Annual Symposium on* (1988), IEEE, pp. 422–431.